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New Properties for Certain Generalized Ces´aro Integral Operator

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***Abstract*— In this work, we obtain the order of convexity of the integral operator which is a generalization to Ces´aro integral operator. Furthermore, some other properties of the integral operator by using the concept of the norm and pre-Schwarzian derivatives are obtained.**

***Keywords*— Analytic function; pre-Shwarzian derivatives; Ces´aro integral operator; starlike function; convex function.**

I. Introduction

The Ces´aro operator C acts formally on the power series $f(z)=\sum_{k=0}^{\infty} a_k(f)z^k$ as

$$C[f](z)=\frac{1}{z}\int_0^z \frac{f(t)}{1-t}\,dt \qquad (1.1)$$

the classical Ces´aro means play an important role in geometric function theory (see [2], [3],[4],[5]).

Let H denote the class of all analytic functions in the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$ of complex plane.

Let A denote the class of functions $f\in H$ normalized by $f(0)=0,\ f'(0)=1$.

Also, let S denote the class of all univalent functions in A.

A function f belonging to A is said to be starlike of order α in U if it satisfies

$$f\in S^*(\alpha) \Leftrightarrow \Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \quad (z\in U),$$

for some $\alpha(0\le\alpha<1)$

Further, a function f belonging to A is said to be convex in U if it satisfies

$$f\in K(\alpha) \Leftrightarrow \Re\left\{\frac{zf''(z)}{f'(z)}+1\right\}>\alpha, \quad (z\in U),$$

for some $\alpha(0\le\alpha<1)$.

A function f belonging to A is said to be the class $R(\alpha)$ iff

$$\Re\{f'(z)\}>\alpha, \quad (z\in U),$$

for some $\alpha(0\le\alpha<1)$.

Very recently, Frasin and Jahangiri [6] defined the family $B(\mu,\alpha)$, for some $(\mu\ge 0, 0\le\alpha<1)$, so that it consists of functions $f\in A$ satisfying the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha, \quad (z\in U) \qquad (1.2)$$

The family $B(\mu,\alpha)$ is a comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well known ones. For example,

$B(1,\alpha)\equiv S^*(\alpha)$, and $B(0,\alpha)\equiv R(\alpha)$.

Another interesting subclass is the special case $B(2,\alpha)\equiv B(\alpha)$, which has been introduced by Frasin and Darus [7] .

Let $f: U\to C$ be analytic and locally univalent. The pre-Schwarzian derivative (or nonlinearity) T_f to f is defined by

$$T_f=\frac{f''}{f'}.$$

Also, with respect to the Hornich operation, the quantity

$$\|T_f\|=\sup_{z\in U}(1-|z|^2)|T_f|,$$

can be regarded as a norm on the space of uniformly locally univalent analytic functions $f \in U$.

It is known that $T_f < \infty$ if and only if f is uniformly locally univalent.

It is well-known that from Becker's univalence criterion [8]: every analytic function f in U with $\|T_f\| \le 1$ is in fact univalent in U .Conversely, $\|T_f\| \le 6$ holds if f univalent.

Consider the general integral operator defined by the formula:

$$C[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z) = \frac{1}{z}\int_0^z (\frac{f_1(t)}{1-t})^{\frac{1}{\beta_1}} ...(\frac{f_m(t)}{1-t})^{\frac{1}{\beta_m}} dt, \ (z \in U\text{-}\{0\}) \ , \ (1.3),$$

where $\beta_i \in \mathbb{C} - \{0\}, \forall i = 1, ..., m$, and the functions $f_i(z)$ are in $B(\mu, \alpha)$. It is clear that when $\beta_1 = 1$ and $\beta_j = 0$, $j = 2, ..., m$ the integral operator (1.3) reduces to Ces´aro integral operator (1.1).

In this paper we will study some general properties for function

$$zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z) = \int_0^z (\frac{f_1(t)}{1-t})^{\frac{1}{\beta_1}} ...(\frac{f_m(t)}{1-t})^{\frac{1}{\beta_m}} dt, (z \in U\text{-}\{0\}) \ .$$

For the purpose this work, we shall make use of the following lemmas.

Lemma 1.1 [1]

Let the analytic function f be regular in the disk with $f(0) = 0$. If $|f(z)| \le 1$, for all $(z \in U)$ then

$|f(z)| \le |z|$, $(z \in U)$.

The equality can hold only if $f(z) = \varepsilon z$,

where $|\varepsilon| = 1$.

Lemma 1.2 Let the analytic and locally univalent f in U . Then

(*i*) If $\|T_f\| \le 1$, then f is univalent , and

(*ii*) If $\|T_f\| \le 2$, then f is bounded .

The part (i) is due to Becker [8] and sharpness of the constant 1 is due to Becker and Pommerenke [9]. The part (ii) is obvious (see [10], Corollary 2.4). Note also that, recently, Kari and Per Hag [12] gave a necessary and sufficient condition for f ∈ S to have a John disk as the image in terms of the preSchwarzian derivative of f .

Also, the norm estimates for typical subclasses of univalent functions are investigated by many authors . See for example ([10], and so on).

Lemma 1.3 [11]

Let $0 \le \alpha < 1$ and $f \in S$.

(i) If f is starlike of order α , then $\|T_f\| \le 6-4\alpha$, and

(*ii*) If f is convex of order α, *then* $\|T_f\| \le 4(1-\alpha)$.

The constants are sharp.

II. Main Results

Theorem 2.1

Let $f_i \in A$, be in the class $B(\mu, \alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$, for all $i = 1, 2, ..., m$. If $|f_i(z)| \le M$, $0 \le |z| < \frac{1}{2}$, $(M \ge 1, z \in U)$, then

$$zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z) = \int_0^z (\frac{f_1(t)}{1-t})^{\frac{1}{\beta_1}} ...(\frac{f_m(t)}{1-t})^{\frac{1}{\beta_m}} dt \ ,$$

is convex of order δ,

where

$$\delta = 1 - \sum_{i=1}^m \frac{1}{|\beta i|}((2-\alpha)M^{\mu-1} + 1),$$

and

$$\sum_{i=1}^m \frac{1}{|\beta i|}((2-\alpha)M^{\mu-1} + 1) < 1, \ \beta_i \in C - \{0\},$$

For all i=1,2,…,m.

Proof:

From the definition of the operator (1.3) ,we have

$$zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z) = \int_0^z \prod_{i=1}^m (\frac{f_i(t)}{1-t})^{\frac{1}{\beta_i}} dt \ ,$$

For $f_i \in B(\mu, \alpha)$. It is easy to see that

$$(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))' = \prod_{i=1}^m (\frac{f_i(t)}{1-t})^{\frac{1}{\beta_i}} \ . \quad (2.1)$$

Differentiating both sides of (2.1) logarithmically, we obtain

$$\frac{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'} = \sum_{i=1}^m \frac{1}{\beta_i}(\frac{f_i'(z)}{f_i(z)} + \frac{1}{1-z}) \ ,$$

which readily shows that

$$\left|\frac{z(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right| \le \sum_{i=1}^m \frac{1}{|\beta_i|}(\left|\frac{zf_i'(z)}{f_i(z)}\right| + \left|\frac{z}{1-z}\right|) \ ,$$

$$= \sum_{i=1}^m \frac{1}{|\beta_i|}(\left| f_i'(z)(\frac{z}{f_i(z)})^{\mu}\right| \left|(\frac{f_i(z)}{z})^{\mu-1}\right| + \left|\frac{z}{1-z}\right|).(2.2)$$

Since $|f_i(z)| \le M, (z \in U,\ i \in \{1, 2, ..., m\})$, applying the Schwarz lemma, we obtain

$$\left|\frac{f_i(z)}{z}\right| \le M, \ (z \in U,\ i \in \{1, 2, ..., m\}).$$

Therefore, from (2.2), we obtain

$$\left|\frac{z(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right| \le$$

$$\sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|M^{\mu-1}+1\right). \qquad (2.3)$$

From (2.3) and (1.2), we see that

$$\left|\frac{z(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right|$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}-1\right|+1\right)M^{\mu-1}+1\right)$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left((2-\alpha)M^{\mu-1}+1\right) \le 1-\delta .$$

This completes the proof.

Theorem 2.2

Let $f_i \in A$, for all $i = 1, 2, ..., m$. Suppose that $zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$ is locally univalent in U.

(1) If $\left[\|T_{f_i}\| + 2\right] \le |\beta_i|$. (2.4)

Then

$zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$, is univalent in U.

(2) If $\left[\|T_{f_i}\| + 2\right] \le 2\,|\beta_i|$. (2.5)

Then

$zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z)$, is univalent in U.

Proof:

Since $\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| = \sup_{z\in U}(1-|z|^2)\left|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right|$. We obtain

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\|$$

$$= \sup_{z\in U}(1-|z|^2)\left|\frac{z(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))''}{(zC[f_1, f_2, ..., f_m]_{\beta_1,\beta_2,...,\beta_m}(z))'}\right|.$$

$$\le \sup_{z\in U}(1-|z|^2)\sum_{i=1}^{m}\frac{1}{|\beta_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+\left|\frac{z}{1-z}\right|\right)$$

$$\le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right].$$

From (2.4),and applying Lemma 1.2 we get

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right] \le 1 .$$

Then $T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}$ is univalent in U.

Also, from (2.5), and applying Lemma 1.2, we get

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le \sum_{i=1}^{m}\frac{1}{|\beta_i|}\left[\|T_{f_i}\| + 2\right] \le 2 .$$

Then $T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}$ is bounded in U.

Theorem 2.3

Let $f_i \in S$, for all $i = 1, 2, ..., m$.

(1) If f_i are starlike of order α_i, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 4\sum_{i=1}^{m}\frac{1}{|\beta_i|}(1-\alpha_i) .$$

(2) If f_i are convex of order α_i, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 2\sum_{i=1}^{m}\frac{1}{|\beta_i|}(3-2\alpha_i) .$$

Proof: The results follow from (2.6) and by using Lemma 1.3.

Corollary 2.1

Let $f_i \in S$, for all $i = 1, 2, \ldots, m$.

(1) If f_i are starlike of order α, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 4(1-\alpha)\sum_{i=1}^{m}\frac{1}{|\beta_i|} .$$

(2) If f_i are convex of order α, then

$$\left\|T_{zC[f_1,f_2,...,f_m]_{\beta_1,\beta_2,...,\beta_m}(z)}\right\| \le 2(3-2\alpha)\sum_{i=1}^{m}\frac{1}{|\beta_i|} .$$

Corollary 2. Let $f_1 \in S$.

(1) If f_1 are starlike of order α, then

$$\left\|T_{zC[f_1]_{\beta_1}(z)}\right\| \le \frac{4(1-\alpha)}{|\beta_1|} .$$

(2) If f_1 are convex of order α, then

$$\left\|T_{zC[f_1]_{\beta_1}(z)}\right\| \le \frac{2(3-2\alpha)}{|\beta_1|} .$$

III. Conclusions

We conclude this study with some suggestions for future research; one direction is to obtain the order of convexity of the integral operator. Another direction would be studying other properties of the integral operator.

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