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Certain Subclasses of Analytic and Bi-Univalent Functions Involving Double Zeta Functions

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***Abstract*— In the present paper, we introduce two new subclasses of the functions class Σ of bi-univalent functions involving double zeta functions in the open unit disc $U=\{z:z\in\mathbb{C},|z|<1\}$. The estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ are obtained in our investigation.**

***Keywords*— Analytic functions, Univalent functions, Bi-univalent functions, Starlike and convex function, Coefficients bounds.**

I. INTRODUCTION

Let A be the class of the function of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \qquad (1)$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. By using the Hadamard product or the convolution product of generalized Hurwitz-Lerch zeta function given by [4], a function is defined as follows:

$$\Psi_n(y,x,a)=\frac{\Phi(y,x,a+\nu n)}{\Phi(y,x,a)} \qquad (2)$$

It is clear that $\Psi_0(y,x,a)=1$. Now consider the function

$$\Upsilon_\mu(z,y,x,a)=\sum_{n=0}^{\infty}\frac{\mu_n}{n!}\Psi_n(y,x,a)z^n \qquad (3)$$

implies

$$z\Upsilon_\mu(z,y,x,a)=z+\sum_{n=2}^{\infty}\frac{(\mu)_{n-1}}{(n-1)!}\Psi_{n-1}(y,x,a)z^n \qquad (4)$$

Thus

$$\Upsilon_\mu(z,y,x,a)*\left(z\Upsilon_\mu(z,y,x,a)\right)^{-1}=\frac{z}{(1-z)^\lambda}, \quad (\lambda>-1)$$

$$=z+\sum_{n=2}^{\infty}\frac{(\lambda)_{n-1}}{(n-1)!}z^n \qquad (5)$$

poses a linear operator

$$I_\mu^\lambda(z,y,x,a)f(z)=\left(z\Upsilon_\mu(z,y,x,a)\right)^{-1}*f(z), \ (f\in A)$$

$$=z+\sum_{n=0}^{\infty}\frac{(\lambda)_{n-1}}{(\mu)_{n-1}\Psi_{n-1}(y,x,a)}a_n z^n \qquad (6)$$

where $|y|<1,|z|<1;\mu\in\mathbb{C}\{\ldots-2,-1,0\},\nu\in\mathbb{C}\{0\}$; $a\in\mathbb{C}\{-(m+\nu n)\}$, $\{nm\}\in\mathbb{N}\cup\{0\}$ and Ψ is defined in (2). It is clear that $I_\mu^\lambda(z,y,x,a)f(z)\in A$. It is based on result by Ibrahim and Darus.

It is well known that every function $f\in S$ has inverse f^{-1}, defined by $f^{-1}(f(z))=z \quad (z\in U)$

and $f(f^{-1}(w))=w \quad (|w|<r_o(f)\geq\frac{1}{4})$

where

$$f^{-1}=w-a_2w^2+(a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots \qquad (7)$$

A function $(f\in A)$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent in U given by the Taylor-Maclaurin series expansion (1). Examples of functions in the class Σ are

$$\frac{z}{(1-z)}, -\log(1-z), \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \qquad (8)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as $z-\frac{z^2}{2}$ and $\frac{z}{1-z^2}$ are also not members of Σ.

Lewin [4] investigated the bi-univalent function class Σ and showed that $|a_2|<1.51$. Subsequently, Brannan and

Clunie [5] conjectured that $|a_2| \le \sqrt{2}$. Netanyahu [6], on the other hand, showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$.

The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients: $|a_n| \quad \left(n \in \mathbb{N}\setminus\{1,2\}; \mathbb{N} := \{1,2,3,\ldots\}\right)$ is presumably still an open problem.

Brannan and Taha [7] (see also [8]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S_*(\kappa)$ and $K(\kappa)$ are starlike and convex function of order $\kappa, (0 \le \kappa < 1)$, respectively (see[9]). Thus, following Brannan and Taha [7] (see also [8]), a function $f \in A$ is in the class $S_\Sigma^*(\alpha)$ $(0 < \alpha \le 1)$ of strongly bi-starlike functions of order α if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \tag{9}$$

$(z \in U; 0 < \alpha \le 1)$ and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2}, \; (w \in U; 0 < \alpha \le 1) \tag{10}$$

where g is the extension of f^{-1} to U. The classes $S_\Sigma^*(\kappa)$ and $K_\Sigma(\kappa)$ of bi-starlike functions of order and bi-convex functions of order κ ,corresponding (respectively) to the function classes $S_*(\kappa)$ and $K(\kappa)$ were also introduced analogously. For each of the function classes $S_\Sigma^*(\kappa)$ and $K_\Sigma(\kappa)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [7,8]).

The object of the present paper is to introduce two new subclasses of the functions class Σ involving double zeta functions operator and find estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ. The techniques of proofing used by Srivastava et. al [4].

II. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathscr{H}_\Sigma^\alpha$

Definition 1. A function f (z) given by (1.1) is said to be in the class $\mathscr{H}_\Sigma^\alpha$ $(0 < \alpha \le 1)$ if the following conditions are satisfied: $f \in \Sigma$ and

$$\left|\arg\left(I_\mu^\lambda(z,y,x,a) f(z)\right)'\right| < \frac{\alpha\pi}{2} \tag{11}$$

$(z \in U; 0 < \alpha \le 1)$ and

$$\left|\arg\left(I_\mu^\lambda(z,y,x,a) g(w)\right)'\right| < \frac{\alpha\pi}{2} \tag{12}$$

$(w \in U; 0 < \alpha \le 1)$ where the function is given by

$$g(w) = w - a_2 w^2 + \left(a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots \tag{13}$$

We first state and prove the following result.

Theorem 1. Let f (z) given by (1) is said to be in the class $\mathscr{H}_\Sigma^\alpha$. Then

$$|a_2| \le \sqrt{\frac{2\alpha^2}{2p_2^2(1-\alpha) + 3p_3\alpha}} \quad \text{and} \quad |a_3| = \frac{\alpha^2}{p_2^2 p_3} + \frac{2\alpha}{3p_3}. \tag{14}$$

Proof. We can write the argument inequalities in (11) and (12) equivalently as follows:

$$\left(I_\mu^\lambda(z,y,x,a) f(z)\right)' = [Q(z)]^\alpha \quad \text{and}$$

$$\left(I_\mu^\lambda(z,y,x,a) g(w)\right)' = [L(w)]^\alpha \tag{15}$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities: $\Re(Q(z)) > 0, (z \in U)$ and $\Re(Q(z)) > 0, (w \in U)$. Furthermore, the functions $Q(z)$ and $L(w)$ have the forms

$$Q(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad \text{and}$$

$$L(w) = 1 + c_1 w + c_2 w^2 + \ldots.$$

Assume,

$$\frac{(\lambda)_{n-1}}{(\mu)_{n-1}\Psi_{n-1}(y,x,a)} = p_n, \tag{16}$$

$$\frac{(\lambda)_1}{(\mu)_1\Psi_1(y,x,a)} = p_2 \tag{17}$$

and

$$\frac{(\lambda)_2}{(\mu)_2\Psi_2(y,x,a)} = p_3 \tag{18}$$

Then, $f(z) = z + \sum_{k=2}^{\infty} p_n a_k z^k$.

Now equating the coefficients of $\left(I_\mu^\lambda(z,y,x,a) f(z)\right)'$ with $[Q(z)]^\alpha$ and the coefficients of $\left(I_\mu^\lambda(z,y,x,a) f(z)\right)'$ with $[L(w)]^\alpha$, we get

$$2p_2 a_2 = \alpha c_1 \tag{19}$$

$$3p_3 a_3 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2 \tag{20}$$

$$-2p_2 a_2 = \alpha l_1 \tag{21}$$

and

$$3p_3\left(2a_2^2 - a_3\right) = \alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2 \tag{22}$$

From (19) and (21), we get

$$c_1 = -l_1 \quad \text{and} \quad 8p_2^2 a_2^2 = \alpha^2 + \left(c_1^2 + l_1^2\right) \tag{23}$$

Also, from (20) and (22), we find that

$$6p_3 a_2^2 - \left(\alpha c_2 + \frac{\alpha(\alpha-1)}{2} c_1^2\right) = \alpha l_2 + \frac{\alpha(\alpha-1)}{2} l_1^2.$$

A rearrangement together with the second identity in (23) yields

$$6p_3 a_2^2 = \alpha(c_2 + l_2) + \frac{\alpha(\alpha-1)}{2}\left(c_1^2 + c_1^2\right)$$

$$= \alpha(c_2 + l_2) + \alpha(\alpha-1)\frac{4p_2^2 a_2^2}{\alpha^2}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2}{4p_2^2(1-\alpha)+6p_3\alpha}(c_2+l_2)$$

which, in conjunction with the following well-known inequalities (see [1, p. 41]): $|c_2| \le 2$ and $|l_2| \le 2$, gives us the desired estimate on $|a_2|$ as asserted in (14).

Next, in order to find the bound on $|a_3|$, by subtracting (22) from (20), we get

$$6p_3a_3 - 6p_3a_2^2 = \alpha c_2 + \frac{\alpha(\alpha-1)}{2}c_1^2 - \left(\alpha l_2 + \frac{\alpha(\alpha-1)}{2}l_1^2\right).$$

Upon substituting the value of a_2^2 from (23) and observing that $c_1^2 = l_1^2$ it follows that

$$a_3 = \frac{\alpha^2 c_1^2}{4p_2^2 p_3} + \frac{\alpha}{6p_3}(c_2 - l_2).$$

The familiar inequalities (see [1, p. 41]): $|c_2| \le 2$ and $|l_2| \le 2$, now yield

$$|a_3| = \frac{\alpha^2}{p_2^2 p_3} + \frac{2\alpha}{3p_3}.$$

This completes the proof of theorem 1.

III. COEFFICIENT BOUNDS FOR THE CLASS $\mathcal{H}_\Sigma(\beta)$

The **Definition 1.** A function $f(z)$ given by (1) is said to be in the class $\mathcal{H}_\Sigma(\beta)$ $(0 \le \beta < 1)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left(I_\mu^\lambda(z,y,x,a)f(z)\right)' > \beta, \quad (24)$$

$(z \in U; 0 \le \beta < 1)$ and

$$\Re\left(I_\mu^\lambda(z,y,x,a)g(w)\right)' > \beta,\ (w \in U; 0 \le \beta < 1). \quad (25)$$

Theorem 2. Let $f(z)$ given by (1) is said to be in the class $\mathcal{H}_\Sigma(\beta)$ $(0 \le \beta < 1)$. Then

$$|a_2^2| \le \frac{2(1-\beta)}{3p_3} \text{ and } |a_3| \le \frac{6p_3}{p_2^2}(1-\beta)^2 + 4(1-\beta) \quad (26)$$

where p_2 and p_3 in (17) and (18), respectively.

Proof. First of all, the argument inequalities in (24) and (25) can easily be rewritten in their equivalent forms:

$$\left(I_\mu^\lambda(z,y,x,a)f(z)\right)' = \beta + (1-\beta)Q(z)$$

and

$$\left(I_\mu^\lambda(z,y,x,a)g(w)\right)' = \beta + (1-\beta)L(w)$$

respectively, where $Q(z)$ and $L(w)$ satisfy the following inequalities: $\Re(Q(z)) > 0, (z \in U)$ and $\Re(Q(z)) > 0, (w \in U)$. Moreover, the functions $Q(z)$ and $L(w)$ have the forms

$$Q(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad (27)$$

and

$$L(w) = 1 + c_1 w + c_2 w^2 + \ldots. \quad (28)$$

As in the proof of Theorem 1, by suitably comparing coefficients, we get

$$2p_2a_2 = (1-\beta)c_1 \quad (29)$$

$$3p_3a_3 = (1-\beta)c_2 \quad (30)$$

$$-2p_2a_2 = (1-\beta)l_1 \quad (31)$$

and

$$3p_3(2a_2^2 - a_3) = (1-\beta)l_2. \quad (32)$$

Now, considering (29) and (31)

$$c_1 = -l_1 \text{ and } 8p_2^2a_2^2 = (1-\beta)^2(c_1^2 + l_1^2) \quad (33)$$

Also, from (30) and (32), we find that

$$6p_3a_2^2 = 3p_3a_3 + (1-\beta)l_2 = (1-\beta)(c_2 + l_2). \quad (34)$$

Therefore, we have

$$|a_2^2| \le \frac{(1-\beta)}{6p_3}(|c_2| + |l_2|) = \frac{2(1-\beta)}{3p_3}. \quad (35)$$

This gives the bound on $|a_2|$ as asserted in (26).

Next, in order to find the bound on $|a_3|$ by subtracting (33) and (30), we get

$$6p_3a_3 - 6p_3a_2^2 = (1-\beta)(c_2 - l_2),$$

which, upon substitution of the value of a_2^2 from (34), yields

$$6p_3a_3 = \frac{6p_3}{8p_2^2}(1-\beta)^2(c_1^2 + l_1^2) + (1-\beta)(c_2 + l_2).$$

This last equation, together with the well-known estimates: $|c_1| \le 2$, $|l_1| \le 2$, $|c_2| \le 2$ and $|l_2| \le 2$.

Lead us to the following inequality:

$$6p_3|a_3| \le \frac{3p_3}{4p_2^2}(1-\beta)^2.8 + (1-\beta).4.$$

Therefore, we have $|a_3| \le \frac{6p_3}{p_2^2}(1-\beta)^2 + 4(1-\beta)$.

This completes the proof of Theorem 2.

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